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The 2D constrained Navier–Stokes equation and intermediate asymptotics

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Abstract
We introduce a modified version of the two-dimensional Navier–Stokes equation, preserving energy and momentum of inertia, which is motivated by the occurrence of different dissipation time scales and is related to the gradient flow structure of the 2D Navier–Stokes equation. The hope is to understand intermediate asymptotics. The analysis we present here is purely formal. A rigorous study of this equation will be done in a forthcoming paper.

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1. Introduction

The simplest model of fluid mechanics is given by the incompressible Euler equation which, by using the vorticity formulation, reads in two dimension:

$$(\partial_t + u \cdot \nabla)\omega(x,t) = 0. \tag{1.1}$$

Here $x \in \mathbb{R}^2$, $t \in \mathbb{R}^+$ and $u = u(x,t) \in \mathbb{R}^2$ is the velocity field defined as

$$u = \nabla^{\perp}\psi, \qquad \psi = -\Delta^{-1}\omega. \tag{1.2}$$

Explicitly, we have

$$u = K * \omega, \qquad K(x) = -\frac{1}{2\pi}\nabla^{\perp}\log|x| = -\frac{1}{2\pi}\frac{x^{\perp}}{|x|^2}.$$

This equation is formally Hamiltonian (we refer, for example, to [MTWR]). The Hamiltonian is the energy

$$E(\omega) = \frac{1}{2}\int \psi\omega\,\mathrm{d}x. \tag{1.3}$$

By Noether theorem, the center of mass $M = \int x\omega$ (which is related to the invariance of E with respect to the group of translations) and the momentum of inertia (which is related to the invariance of E with respect to the group of rotations) $I(\omega) = \frac{1}{2}\int |x - M|^2\omega\,\mathrm{d}x$ are also conserved. Moreover, due to the degeneracy of the Poisson bracket, one of the main features of (1.1) is the presence of an infinite number of conserved quantities, sometimes called Casimir: all the integrals of the form

$$F_\phi(\omega) = \int \phi(\omega)\,\mathrm{d}x$$

are conserved. For this reason, the rigorous study of the large-time behavior of the solutions of (1.1) and hence the justification from (1.1) of the presence of coherent structures observed in real and numerical experiments (see e.g. [MS]) remains widely open. In the following, we shall focus on non-negative solutions and shall assume that ω is a probability distribution so that ω is non-negative and $\int \omega = 1$ constantly in time. We also fix the reference frame in such a way that $\int x\omega = 0$ for every time.

One attempt to justify the appearance of these coherent structures is due to Onsager [O], see also [LP, MJ]. The main idea is to replace the Euler equation (1.1) by the system of N point vortices and to study the statistical mechanics of these point vortices. In a limit $N \to +\infty$, the Gibbs measure associated with the point vortices concentrates on some special stationary solutions of the Euler equation (called mean-field solutions); this was rigorously justified in [CLMP1, CLMP2, K, KJ]. These states are under the form:

$$\omega = \frac{\mathrm{e}^{b\psi + a\frac{|x|^2}{2}}}{Z}, \tag{1.4}$$

where

$$Z = \int \mathrm{e}^{b\psi + a\frac{|x|^2}{2}}$$

is a normalization to have $\int \omega = 1$. Recalling that $\omega = -\Delta\psi$, we realize that equation (1.4) is a nonlinear elliptic equation. The study of this equation in connection with the variational principles arising from statistical mechanics was performed in [CLMP1, CLMP2]. Let us summarize the main results. We define the free-energy functional as

$$F_{(b,a)}(\omega) = S(\omega) - bE(\omega) - aI(\omega) \tag{1.5}$$

for a given pair $a < 0$ and $b > 0$. $F_{(b,a)}$ is defined on the space Γ of all the probability densities on $\mathbb{R}^2$ with finite entropy, energy and moment of inertia. We define the canonical variational principle as

$$F(b,a) = \inf_{\omega\in\Gamma} F_{(b,a)}(\omega). \tag{1.6}$$

Next, for $E \in \mathbb{R}$ and $I > 0$ let us introduce the set

$$\Gamma_{(E,I)} = \{\omega \in \Gamma \mid E(\omega) = E, I(\omega) = I\} \tag{1.7}$$

and consider the microcanonical variational principle

$$S(E,I) = \inf_{\omega\in\Gamma_{(E,I)}} S(\omega). \tag{1.8}$$

The above variational problems can be related to the solutions of the mean-field equation (1.4). Moreover, in the whole space $\mathbb{R}^2$, we have also the equivalence of the ensembles:

Theorem 1 [CLMP1, CLMP2]**.** *For $a < 0$ and $0 < b < 8\pi$:*

(i) There exists a unique, radially symmetric minimizer $\omega = \omega_{(b,a)} \in \Gamma$ of the problem (1.6) which is the unique radially symmetric solution to equation (1.4).

(ii) When $b \to 8\pi$, ω converges (weakly) to a δ at the origin.

(iii) $F(b, a)$ is a concave smooth function and

$$\frac{\partial F}{\partial a} = I(\omega_{(b,a)}), \qquad \frac{\partial F}{\partial b} = -E(\omega_{(b,a)})$$

(iv) For $E \in \mathbb{R}$ and $I > 0$ define

$$S^*(I, E) = \sup_{a,b}(F(b, a) + bE + aI)$$

and denote by $b(I, E)$ and $a(I, E)$ the unique maximizers. Then $S(I, E) = S^*(I, E)$ and hence S is a smooth convex function.

(v) The variational problem (1.8) has a unique minimizer $\omega(I, E)$ and

$$\omega(I, E) = \omega_{(b(I,E),a(I,E))}.$$

Note that when $b \leqslant 0$ the theory is easier. Indeed, the functional $F_{(b,a)}(\omega)$ is convex so the minimization problem is standard and equation (1.4) has a unique (radial) solution [GL]. We also point out that the equivalence between (1.6) and (1.8) which is established in (v) is also useful for establishing the existence of a minimizer for (1.8). Indeed, in an unbounded domain, the existence of a minimizer for (1.8) seems difficult to establish directly because of the absence of higher moment control which would allow us to pass to the limit in the constraint for the moment of inertia. Also note that equation (1.4) has a natural statistical mechanical interpretation, its solutions being Gibbs states with a self-consistent interaction. Therefore $-b$ is an inverse temperature. Hence $b > 0$ implies negative temperature states, as predicted by Onsager [O] in terms of point vortex theory.

A rigorous justification of the fact that the solutions of the mean-field equation play a special part in the large-time behavior of the Euler equation still seems an out-of-reach problem. An attempt toward the justification of the fact that the states (1.4) play a special part in the 2D turbulence could come from the study of the intermediate behavior of the Navier–Stokes equation. Indeed, for the Navier–Stokes equation,

$$(\partial_t + u \cdot \nabla)\omega(x, t) = \nu\Delta\omega(x, t), \tag{1.9}$$

$\int \omega$ and $\int x\omega$ are still conserved so that we can still consider non-negative solutions so that $\int \omega = 1$ and $\int x\omega = 0$. Nevertheless, due to the dissipation term on the right-hand side of equation (1.9), the asymptotic behavior of the solutions is trivial, namely $\omega(x, t) \to 0$ pointwise and in the L^p sense for $p > 1$. Consequently, one can hope to observe the mean-field solutions only as intermediate states. To formalize this idea, let us note that the momentum of inertia I increases by a constant rate:

$$\dot{I}(\omega) = 2\nu, \tag{1.10}$$

consequently it can be considered as constant for times $\ll 1/\nu$. In a similar way, E and S are dissipated with the rates

$$\dot{E} = -\nu \int \omega^2, \qquad \dot{S} = -\nu \int \frac{|\nabla\omega|^2}{\omega}. \tag{1.11}$$

Looking at equations (1.11), one realizes that the energy could also evolve from a different and longer scale of times with respect to S (whenever the last term in (1.11) dominates the first one). This would suggest considering, in the first approximation, E and I as constant, by looking at a master equation which modifies the Navier–Stokes equation leaving constant both energy and moment of inertia, but retaining all the other features of the Navier–Stokes dynamics. The derivation of such an equation based on geometric arguments is the aim of the following section.

2. Derivation of the model

A natural and fruitful way to approach the problem is to invoke a recent characterization of the Navier–Stokes equation connected with the mass transport problem and the associated differential calculus introduced in [Ot]. We follow the excellent monographies [V, AGS] for outlining the main ideas.

Let $\mathcal{M}$ be the manifold of the probability measures on $\mathbb{R}^2$. One can formally give to $\mathcal{M}$ a structure of Riemannian manifold. For any $\rho \in \mathcal{M}$ we parametrize the tangent space as

$$T_\rho\mathcal{M} = \{\dot\rho \mid \dot\rho = -\mathrm{div}\, u\}. \tag{2.1}$$

Equation (2.1) expresses the tangent space to any point ρ of $\mathcal{M}$ as mass preserving velocity vectors $\dot\rho$. Next, we define a Riemannian metric. On the tangent space $T_\rho\mathcal{M}$, we define a scalar product by

$$\langle \dot\rho_1, \dot\rho_2\rangle_W = \int \rho^{-1} u_1 \cdot u_2 \,\mathrm{d}x, \tag{2.2}$$

being $\dot\rho_i = -\mathrm{div}\, u_i$, $i = 1, 2$. The gradient ∇_W with respect to this Riemannian metric of a functional $F : \mathcal{M} \to \mathbb{R}$ is defined as

$$\langle \nabla_W F, \dot\rho\rangle_W = DF \cdot \dot\rho = -\int \frac{\delta F}{\delta\rho}\,\mathrm{div}\, u,$$

where DF is the differential of the map F and $\delta F/\delta\rho$ is the usual variational derivative. An explicit computation shows that

$$\nabla_W F = -\mathrm{div}\left[\rho\nabla\frac{\delta F}{\delta\rho}\right]. \tag{2.3}$$

In a similar way, we can define on $T_\rho\mathcal{M}$ a skew-symmetric operator J_W by

$$J_W\dot\rho = -\mathrm{div}(u^\perp), \qquad \dot\rho = -\mathrm{div}\, u \tag{2.4}$$

where $u^\perp = (u_2, -u_1)$. Note that this yields in particular the expression

$$J_W\nabla_W F = -\mathrm{div}\left[\rho\nabla^\perp\frac{\delta F}{\delta\rho}\right]. \tag{2.5}$$

Gradient flows with respect to a functional F are the solutions to

$$\partial_t\rho = -\nabla_W F \tag{2.6}$$

and are dissipative in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t}F = -\int \omega\left|\nabla\frac{\delta F}{\delta\rho}\right|^2 \leqslant 0$$

whereas Hamiltonian flows defined by

$$\partial_t\rho = J_W\nabla_W F$$

are conservative since

$$\frac{\mathrm{d}}{\mathrm{d}t}F = 0.$$

Since $\delta E/\delta\omega = \psi$, we get from (2.5) that

$$\mathrm{div}[\omega\nabla^\perp\psi] = J_W\nabla_W E$$

and hence, the Euler equation can be interpreted as a Hamiltonian flow in this framework.

Next, to interpret the dissipative part of the Navier–Stokes equation, we note that

$$\Delta\omega = \mathrm{div}\left(\omega\nabla\frac{\delta S}{\delta\omega}\right) = -\nabla_W S,$$

where S is the entropy functional.

In conclusion the Navier–Stokes equation can be expressed in terms of a gradient and an antigradient flow:

$$\partial_t\omega = -\nu\nabla_W S + J_W\nabla_W E. \tag{2.7}$$

According to the previous discussion we assume that the energy and the moment of inertia are varying much more slowly than the entropy functional. Therefore it may be useful to derive an effective equation according to the following prescription. We shall take the orthogonal projection (with respect to the scalar product (2.2)) of the vector field on the right-hand side of equation (2.7) on the manifold $E = \text{const}$ and $I = \text{const}$ with the aim of characterizing the states which are close to the true dynamics for a large interval of times (coherent structures), as asymptotic states of the new dynamics. We shall see that such states are the solutions to the mean-field equation (1.4). Since

$$\nabla_W E(\omega) = -\mathrm{div}\left[\omega\nabla\frac{\delta E(\omega)}{\delta\omega}\right] = -\mathrm{div}[\omega\nabla\psi],$$

$$\nabla_W I(\omega) = -\mathrm{div}\left[\omega\nabla\frac{\delta I(\omega)}{\delta\omega}\right] = -\mathrm{div}\left[\omega\nabla\frac{x^2}{2}\right] = -\mathrm{div}[\omega x]$$

we find that

$$\langle\nabla_W E, J_W\nabla_W E\rangle_W = 0 \qquad \langle\nabla_W I\cdot J_W\nabla_W E\rangle_W = 0,$$

and hence $J_W\nabla_W E$ is tangent to the manifold $E = \text{const}$, $I = \text{const}$. On the other hand, the projection of $\nabla_W S$ on the tangent space of such a manifold is of the form

$$\nabla_W S - b\,\mathrm{div}(\omega\nabla\psi) - a\,\mathrm{div}(\omega x),$$

with a and b two suitable multiplicators. As a consequence, the equation we are looking for is:

$$\partial_t\omega + u\cdot\nabla\omega = \nu\mathrm{div}(\nabla\omega - b\omega\nabla\psi - a\omega x) = \nu\,\mathrm{div}\left[\omega\nabla\left(\log\omega - b\psi - a\frac{x^2}{2}\right)\right], \tag{2.8}$$

with a and b to be determined by the simultaneous conservation of E and I. A straightforward computation yields

$$b = \frac{2I\int\omega^2 + 2V}{2I\int\omega|\nabla\psi|^2 - V^2}, \qquad a = -\frac{2\int\omega|\nabla\psi|^2 + V\int\omega^2}{2I\int\omega|\nabla\psi|^2 - V^2}, \tag{2.9}$$

where

$$V = \int\omega x\cdot\nabla\psi = \int\mathrm{d}x\int\mathrm{d}y\,\omega(x)\omega(y)x\cdot\nabla g(x-y) = -\frac{1}{4\pi}. \tag{2.10}$$

Note that we have by the Cauchy–Schwarz inequality

$$V^2 = \left(\int\omega x\cdot\nabla\psi\right)^2 \leqslant 2I\int\omega|\nabla\psi|^2, \tag{2.11}$$

and hence b is positive if inequality (2.11) holds strictly and $\int\omega^2 > \frac{1}{4\pi I}$.

We point out that equation (2.8) has been derived in [Ch2] (see also [Ch1]), by using different arguments than those of the present paper.

Also note that another class of similar equations was introduced in [RS] and studied mathematically in [MR]. Nevertheless, despite some formal analogy the mathematical properties of the equations studied in [MR] are very different from those presented here.

Let us now discuss what we may expect about the asymptotic behavior of equation (2.8). The main feature of equation (2.8) is the decay of the entropy functional. Indeed, we have

$$\begin{aligned}\frac{\mathrm{d}S(\omega)}{\mathrm{d}t} &= \frac{\mathrm{d}S(\omega)}{\mathrm{d}t} - b\frac{\mathrm{d}E(\omega)}{\mathrm{d}t} - a\frac{\mathrm{d}I(\omega)}{\mathrm{d}t}\\ &= \nu\int\left(\frac{\delta S}{\delta\omega} - b\frac{\delta E}{\delta\omega} - a\frac{\delta I}{\delta\omega}\right)\mathrm{div}\left[\omega\nabla\left(\frac{\delta S}{\delta\omega} - a\frac{\delta I}{\delta\omega} - b\frac{\delta E}{\delta\omega}\right)\right]\\ &= -\nu\int\omega\left|\nabla\left(\frac{\delta S}{\delta\omega} - a\frac{\delta I}{\delta\omega} - b\frac{\delta E}{\delta\omega}\right)\right|^2\\ &= -\nu\int\omega\left|\nabla(\log\omega - b\psi - a\frac{|x|^2}{2}\right|^2.\end{aligned}$$

In particular, this suggests formally that the asymptotic states satisfy

$$\omega\nabla\left(\log\omega - b\psi - a\frac{|x|^2}{2}\right) = 0$$

and hence the mean-field equation (1.4).

Finally, let us point out that the procedure for constructing dissipative equations leaving invariant a given quantity is not unique. For instance the heat equation in $\mathbb{R}^2$:

$$\partial_t\omega = \Delta\omega$$

can be modified to leave invariant I according to the procedure suggested by the gradient flow structure. The result is

$$\partial_t\omega = \mathrm{div}(\nabla\omega - a\omega x), \tag{2.12}$$

where

$$a = -\frac{1}{I}.$$

On the other hand we could also have

$$\partial_t\omega = \partial^2_{\theta,\theta}\omega. \tag{2.13}$$

Note that equations (2.12) and (2.13) have different asymptotic states.

3. The Oseen vortex

An attempt to characterize an intermediate asymptotics was presented by Gallay and Wayne [GW] according to the following ideas.

It is well known that a special solution to equation (1.9) (for $\nu = 1$) is given by the so-called Oseen vortex:

$$\omega(x, t) = \frac{1}{4\pi(t+1)}\,\mathrm{e}^{-\frac{|x|^2}{4(t+1)}}.$$

Note that this is also a solution to the heat equation. It was shown in [GW] that this solution describes the long time asymptotic of the Navier–Stokes equation in L^1. Indeed, with the change of variables

$$\xi = \frac{x}{\sqrt{1+t}}, \qquad \tau = \log(1+t), \qquad \omega(x, t) = (1+t)^{-1}w(\xi, \tau),$$

the Navier–Stokes equation in the new variables is under the form

$$\partial_\tau w + v \cdot \nabla_\xi w = \Delta_\xi w + \nabla_\xi \cdot \left(\tfrac{1}{2}\xi w\right). \tag{3.1}$$

It is possible to show that $w \to W$ in L^1 as $t \to \infty$, where $W(\xi)$ is the rescaled Oseen vortex. As a consequence the Oseen vortex can be thought of as characterizing an intermediate asymptotics before the dissipation scale. Note that W is also a solution to (1.4) for $b = 0$.

This analysis enters perfectly in the context of the projected gradient flows. Indeed neglecting the energy, the mere constance of I yields

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \frac{1}{I} \nabla \cdot (\omega x), \tag{3.2}$$

that is equation (3.1) for $I = 2$. In this particular case we have seen how equation (3.1) can also be obtained by a simple change of variables, due to the fact that the dissipation rate of I is constant. Roughly speaking, the analysis we present here is an attempt to see what happens before the appearance of the Oseen vortex by projecting on a less robust manifold. Indeed, one could argue that I is more stable than E in many interesting physical situations. If so equation (2.8) should be more appropriate on the time scale when E is practically constant, while equation (3.1) should describe the fluid when E starts to be dissipated at constant I. After that, everything disappears.

It would be interesting to look for numerical evidence of this fact, if true.

4. Derivation of (2.8) by using stochastic vortices

The formal derivation of equation (2.8) is also in agreement with the stochastic vortex theory as we are going to illustrate. As explained previously, the procedure we follow for constructing a dissipative equation leaving invariant the energy and the moment of inertia, although quite reasonable, is not unique. Therefore, it is interesting to observe that one gets the same equation by using other arguments. Equation (2.8) has been interpreted as a constrained Navier–Stokes flow and it turns out that the asymptotic states are the solution of the microcanonical variational principle. Since this variational principle is obtained by the mean-field limit from the statistical theory of the point vortices, it is interesting to see how equation (2.8) is related to the theory of point vortices. Now we show how, at least formally, the structure of the constrained gradient flow is compatible with the discretization of the Navier–Stokes equation obtained by means of the stochastic vortex theory. Since in this context we are not interested in the asymptotic behavior, we limit ourselves to considering the energy constraint only. Also the viscosity does not play any role so we set $\nu = 1$.

Consider N stochastic vortices in $\mathbb{R}^2$. They obey the stochastic differential equation:

$$\mathrm{d}x_i = \frac{1}{N} \sum_j \nabla^\perp g(x_i - x_j)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}w_i, \tag{4.1}$$

where $\{w_i\}_{i=1}^N$ are N independent standard Brownian motions. Here $g(x)$ is a regularization of the Green function $-\frac{1}{2\pi}\log|x|$. It is well known ([MP2, Os] ...) that the empirical random measure

$$\mu_N(\mathrm{d}x, t) = \frac{1}{N} \sum_{j=1}^N \delta(x_j(t) - x)\,\mathrm{d}x \tag{4.2}$$

approaches the solution to the (regularized) Navier–Stokes equation, if it happens at time zero.

We now consider the mean-field energy:

$$H(x_1 \dots x_N) = \frac{1}{N} \sum_{j<r} g(x_j - x_r).$$

In order to guarantee the condition

$$\mathrm{d}H = 0$$

a short computation using the Ito formula shows that we have to modify the process according to

$$\mathrm{d}x_i = \frac{1}{N}\sum_j [\nabla^\perp g(x_i - x_j) - b_N(t)\nabla g(x_i - x_j)]\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}w_i$$
$$+ \sum_{i,j} D^2_{i,j} H \frac{\nabla_i H \cdot \nabla_j H}{|\nabla H|^4}\,\mathrm{d}t - \frac{1}{|\nabla H|^2}\sum_j \nabla_j H \cdot \mathrm{d}w_j \nabla_i H, \tag{4.3}$$

where

$$b_N(t) = \frac{\Delta H}{|\nabla H|^2} = \frac{\int \mathrm{d}\mu_N \Delta g * \mu_N}{\int \mathrm{d}\mu_N |\nabla g * \mu_N|^2}.$$

Note that the above expression makes sense only if we regularize g and this explains why we did it.

We observe that an analysis on the size of the last two terms in (4.3) shows that they should be negligible in the limit $N \to \infty$. Thus we introduce the essential process defined by

$$\mathrm{d}y_i = \frac{1}{N}\sum_j [\nabla^\perp g(y_i - y_j) - b_N(t)\nabla g(y_i - y_j)]\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}w_i.$$

Now if we replace $\mu_N(\mathrm{d}x, t)$ by $\omega(x, t)\,\mathrm{d}x$ in the limit $N \to \infty$, each process x_j approaches the nonlinear (in the McKean sense, see [Mc, MP2, Os]) process solution to

$$\mathrm{d}y = [\nabla^\perp g * \omega(y) - b(t)\nabla g * \omega(y)]\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}w, \tag{4.4}$$

where

$$b(t) = \frac{\int \mathrm{d}x\, \omega \Delta g * \omega}{\int \mathrm{d}x\, \omega |\nabla g * \omega|^2}.$$

From equation (4.4), we also derive the backward Kolmogorov equation for the probability distribution ω of the process y:

$$\partial_t \omega + \nabla \cdot (u\omega) = \nabla \cdot (\nabla\omega - b\omega\nabla g * \omega), \tag{4.5}$$

where $u = \nabla^\perp g * \omega$. It is remarkable that such an equation is the microcanonical equation (2.8), for $a = 0$, if we replace g by the true Green function in equation (4.5).

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